Equilibrium Flow Shear Implementation in GS2

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1 Notes on Toroidal Flow Shear in GK

1.1 CMR Notes on including Toroidal Flow Shear in GK

In the absence of toroidal flow, the linear electrostatic gyrokinetic equation is derived for the perturbed distribution function for an isotropic equilibrium:

$$f_0^1 = n(x)\left(\frac{m}{2\pi T(x)}\right)^{1.5} e^{-mv^2/2T(x)}$$

where $n(x)$ and $T(x)$ represent the equilibrium temperature and density on a given flux surface labelled by x. The perturbed distribution function at next order is given by:

$$f_1^1 = \frac{q\Phi_1}{m}\frac{\partial f_0^1}{\partial E} + g(\boldsymbol{r}, v_\parallel, v_\perp)e^{-i\boldsymbol{k}.\boldsymbol{\rho}}. \quad (1)$$

where the nonadiabatic part of the perturbed distribution function g is obtained from the gyrokinetic equation:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{U}.\boldsymbol{\nabla}\right)g + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v_d}\right).\boldsymbol{\nabla} g = -\frac{qJ_0(Z)}{m}\frac{\partial f_0^1}{\partial E}\left(\frac{\partial}{\partial t} + \boldsymbol{U}.\boldsymbol{\nabla}\right)\Phi_1 + \frac{J_0(Z)}{B}\boldsymbol{\nabla}\Phi_1 \times \boldsymbol{b}.\boldsymbol{\nabla} f_0^1 \quad (2)$$

where $Z = k\rho$, $\boldsymbol{U}$ is a mean flow and the velocity of a particle guiding centre can be written as:

$$\boldsymbol{v} = \boldsymbol{U} + v_\parallel \boldsymbol{b} + \boldsymbol{v_d}.$$

Now we shall assume that the equilibrium distribution function has subsonic sheared toroidal rotation that can be represented in the equilibrium distribution function by:

$$f_0^1 = n(x)\left(\frac{m}{2\pi T(x)}\right)^{1.5} e^{-m\left(\boldsymbol{v} - R\Omega(x)\boldsymbol{e_\phi}\right)^2/2T(x)} = n(x)\left(\frac{m}{2\pi T(x)}\right)^{1.5} e^{-mv^2/2T(x) + mv_\phi R\Omega(x)/T(x)}$$

where x is a flux surface label proportional to poloidal flux. Now if we assume low Mach number:

$$R\Omega(x) \sim O\left(\varepsilon v_t\right)$$

where $\varepsilon \ll 1$ so that the impact of toroidal rotation on the plasma equilibrium (via Coriolis and centrifugal forces) can be neglected, and transform to the frame that co-rotates toroidally with the surface labelled by x_0, denoting $x - x_0$ by x we can write the equilibrium distribution function as:

$$\Rightarrow f_0^1 \quad = \quad n(x_0)\left(\frac{m}{2\pi T(x_0)}\right)^{1.5} e^{-mv^2/2T(x_0)}\left(1 + \frac{mv_\phi R\Omega'(x_0)x}{T(x_0)}\right) \quad (3)$$

where $'$ denotes a derivative with respect to poloidal flux d/dx. In order to resolve equilibrium flow shear stabilisaton, we require that the equilibrium flow shear is of the order of mode growth rate: ie $R^2B_p\Omega' \sim O(v_t/L)$. This corresponds to a gradient scale length for the toroidal frequency $L_\Omega \sim O(\varepsilon L)$. The change in toroidal flow velocity ΔV_ϕ across a flux-tube domain of radial width $\Delta r \sim O(\rho)$ is therefore given by $\Delta V_\phi \sim O(v_t\rho/L)$, so that the toroidal flow remains subsonic across the domain. Thus the second term in the bracket above is a higher order ρ/L correction to the equilibrium distribution function. Nevertheless when equilibrium flow shear is comparable with growth rates, the radial derivative of this second term is comparable to the leading order equilibrium gradients that appear on the RHS of the gyrokinetic equation, and so the derivative of this term should be included there.

The change in electrostatic potential $\Delta\Phi$ across the narrow flux-tube associated with the sheared flow is given by

$$\Delta\Phi = \Phi' R B_p \rho + \Phi'' R^2 B_p^2 \rho^2 \sim O\left(\varepsilon + \frac{\rho}{L}\right)\frac{T}{q}$$

Now

$$\Phi' = \Omega \sim O(\varepsilon \frac{v_t}{L}) \text{ and } \Phi'' = \Omega' = O\left(\frac{1}{R^2 B_p}\frac{v_t}{L}\right)$$

so that

$$\frac{q\Delta\Phi}{T} = O\left(\frac{\rho}{\rho_p}\left(\varepsilon + \frac{\rho}{L}\right)\right)$$

and that the change in electrostatic potential across the tube is a small fraction of the particle kinetic energy.

The terms in (2) where $(\partial/\partial t + \boldsymbol{U}.\boldsymbol{\nabla})$ acts on perturbation quantities can be represented as time derivatives with a time dependent eikonal, and this will be discussed shortly.

The second term on the RHS of equation (2) includes a radial derivative of the toroidal rotation frequency, which contributes to leading order the following additional term T_R:

$$T_R = \frac{m v_\phi R\Omega'}{T} f_0^1 \frac{J_0(Z)}{B} \boldsymbol{\nabla}\Phi_1 \times \boldsymbol{b}.\boldsymbol{\nabla} x$$

Given that the variation in the equilibrium electrostatic potential across the flux-tube domain satisfies $q\Delta\phi/T \sim O\left(\frac{\rho}{\rho_p}\left(\varepsilon + \frac{\rho}{L}\right)\right)$ throughout the flux-tube domain, we can approximate for v_ϕ using:

$$v_\phi = v_\parallel \frac{B_\phi}{B} = \sqrt{\frac{2(E - \mu B)}{m}}\frac{B_\phi}{B}$$

where $E = \frac{mv^2}{2}$ and $\mu = \frac{mv_\perp^2}{2B}$, and we have dropped the higher order contribution $R\Omega' x$. The additional term T_R can be expressed as:

$$T_R = \frac{m v_\parallel R\Omega'}{T}\frac{B_\phi}{B} f_0^1 \frac{J_0(Z)}{B} \boldsymbol{\nabla}\Phi_1 \times \boldsymbol{b}.\boldsymbol{\nabla} x \tag{4}$$

The gyrokinetic equation with flow shear becomes:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \boldsymbol{U}.\boldsymbol{\nabla}\right) g + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v_d}\right).\boldsymbol{\nabla} g = & -\frac{qJ_0(Z)}{m}\frac{\partial f_0}{\partial E}\left(\frac{\partial}{\partial t} + \boldsymbol{U}.\boldsymbol{\nabla}\right)\Phi_1 + \frac{J_0(Z)}{B}\boldsymbol{\nabla}\Phi_1 \times \boldsymbol{b}.\boldsymbol{\nabla} f_0 \\ & + \frac{m v_\parallel R\Omega'}{T}\frac{B_\phi}{B} f_0 \frac{J_0(Z)}{B}\boldsymbol{\nabla}\Phi_1 \times \boldsymbol{b}.\boldsymbol{\nabla} x \end{aligned} \tag{5}$$

where $f_0 = n(x)\left(\frac{m}{2\pi T(x)}\right)^{1.5} e^{-mv^2/2T(x)}$ does not include the toroidal equilibrium flow.

GS2 presently includes the time dependent eikonal, but does not include the term T_R that should appear on the RHS of the gyrokinetic equation if the equilibrium distribution function is self-consistent with the sheared equilibrium toroidal flow.

1.2 GKE with Subsonic Flow Shear from Artun and Tang

The nonlinear gyrokinetic equation with up to sonic toroidal flows is given in equation (56) of Artun and Tang, Physics of Plasmas **1**, 2682 (1994). Dropping magnetic perturbations and nonlinear terms for brevity, A&Ts' equation (56) can be written as:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \left(c_\parallel \boldsymbol{b} + \boldsymbol{V} + \boldsymbol{c_D}\right).\boldsymbol{\nabla}_R\right)\delta g \;\; = & \;\; -\frac{qJ_0}{m}\frac{\partial F}{\partial \epsilon}\left(\frac{\partial}{\partial t} + \boldsymbol{V}.\boldsymbol{\nabla}_R\right)\tilde{\Phi} - \frac{qJ_0}{m\Omega}\boldsymbol{b}\times\boldsymbol{\nabla}_R\tilde{\Phi}.\boldsymbol{\nabla}_R F \\ & + \frac{qJ_0}{m\Omega}\frac{\partial F}{\partial \epsilon}\boldsymbol{b}\times\boldsymbol{\nabla}_{\mathrm{R}}\tilde{\Phi}.\left[\left(c_\parallel \boldsymbol{b} + \boldsymbol{V}\right).\boldsymbol{\nabla}\boldsymbol{V} + \boldsymbol{\nabla}\boldsymbol{V}.\left(c_\parallel \boldsymbol{b} + \boldsymbol{V}\right)\right]\end{aligned} \tag{6}$$

where the velocity of a particle guiding centre $\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{c}$, with $\boldsymbol{V}$ representing the mean toroidal flow $\boldsymbol{V} = R\Omega(\psi)\boldsymbol{e_\phi}$, and with $\boldsymbol{c} = c_\parallel \boldsymbol{b} + \boldsymbol{c_D}$. The guiding centre drifts are contained in $\boldsymbol{c_D}$:

$$\boldsymbol{c_D} = \frac{\boldsymbol{b}}{\Omega}\times\left(\frac{q}{m}\boldsymbol{\nabla}\phi_0 + \frac{c_\perp^2}{2}\boldsymbol{\nabla}\log B + c_\parallel^2 \boldsymbol{b}.\boldsymbol{\nabla}\boldsymbol{b} + 2c_\parallel \boldsymbol{b}.\boldsymbol{\nabla}\boldsymbol{V} + \boldsymbol{V}.\boldsymbol{\nabla}\boldsymbol{V}\right)$$

which includes the $\boldsymbol{E}\times\boldsymbol{B}$ drift, the $\boldsymbol{\nabla} B$ drift, the magnetic curvature drift, and the Coriolis and centrifugal drifts (that arise in the sheared co-rotating frame adopted for this calculation) respectively. The leading order distribution function is given by $F = F(\epsilon, \mu, R_\perp)$, where the particle energy $\epsilon_0 = c_\|^2/2 + c_\perp^2/2 - V^2/2 + q\phi_0/m$ and magnetic moment $\mu_0 = c_\perp^2/2B$.

Here we proceed to simplify (6) to give a gyrokinetic equation with subsonic sheared flows. We assume that $V = R\Omega(\psi) \sim O(\varepsilon v_t)$, while allowing a locally steep radial gradient in the toroidal flow frequency so that $\partial\boldsymbol{V}/\partial x = R^2 B_p \Omega' \boldsymbol{e_\phi} = O(v_t/L)$ (ie $L_\Omega \sim O(\varepsilon L)$. We restrict $\boldsymbol{V}$ further to consider only the situation with only sheared toroidal flow by writing $\boldsymbol{V} = R\Omega' x \boldsymbol{e_\phi}$, where in the local equilibrium $x = \psi - \psi_0$ and the toroidal flow is zero at $x = 0$. With these assumptions we have that:

$$
\begin{aligned}
\boldsymbol{b}.\boldsymbol{\nabla}\boldsymbol{V} &= \Omega(\psi)\boldsymbol{b}.\boldsymbol{\nabla}(R\boldsymbol{e_\phi}) = \Omega(\psi)\left(-\frac{B_\phi}{B}\boldsymbol{e_R} + \frac{\boldsymbol{B_p}.\boldsymbol{\nabla}\theta}{B}\frac{\partial R}{\partial\theta}\boldsymbol{e_\phi}\right) \sim O\left(\frac{\varepsilon v_t}{L}\right) \\
\boldsymbol{V}.\boldsymbol{\nabla}\boldsymbol{V} &= -R\Omega^2\boldsymbol{e_R} \sim O\left(\frac{\varepsilon^2 v_t^2}{L}\right) \\
\boldsymbol{\nabla}\boldsymbol{V}.\boldsymbol{b} &= \boldsymbol{\nabla}(R\Omega)\boldsymbol{e_\phi}.\boldsymbol{b} + R\Omega\cancelto{0}{\boldsymbol{\nabla}\boldsymbol{e_\phi}.\boldsymbol{b}} \sim \frac{B_\phi}{B}R\Omega'\boldsymbol{\nabla}\psi \sim O\left(\frac{v_t}{L}\right) \\
\boldsymbol{\nabla}\boldsymbol{V}.\boldsymbol{V} &= R\Omega\boldsymbol{\nabla}R\Omega + R^2\Omega^2\cancelto{0}{\boldsymbol{\nabla}\boldsymbol{e_\phi}.\boldsymbol{e_\phi}} \sim R^2\Omega\Omega'\boldsymbol{\nabla}\psi \sim O\left(\frac{\varepsilon v_t^2}{L}\right)
\end{aligned}
$$

With these orderings, to leading order in ε we can drop the Coriolis and centrifugal drifts:

$$
\boldsymbol{c_D} = \boldsymbol{c_{D0}} = \frac{\boldsymbol{b}}{\Omega}\times\left(\frac{q}{m}\boldsymbol{\nabla}\phi_0 + \frac{c_\perp^2}{2}\boldsymbol{\nabla}\log B + c_\|^2\boldsymbol{b}.\boldsymbol{\nabla}\boldsymbol{b}\right)
$$

and we find that:

$$
\left[\left(c_\|\boldsymbol{b} + \boldsymbol{V}\right).\boldsymbol{\nabla}\boldsymbol{V} + \boldsymbol{\nabla}\boldsymbol{V}.\left(c_\|\boldsymbol{b} + \boldsymbol{V}\right)\right] = c_\|\boldsymbol{\nabla}\boldsymbol{V}.\boldsymbol{b} = c_\| R\Omega'\frac{B_\phi}{B}\boldsymbol{\nabla}\psi
$$

Therefore equation (6) reduces to:

$$
\begin{aligned}
\left(\left(\frac{\partial}{\partial t} + \boldsymbol{V}.\boldsymbol{\nabla}_R\right) + \left(c_\|\boldsymbol{b} + \boldsymbol{c_{D0}}\right).\boldsymbol{\nabla}_R\right)\delta g &= -\frac{qJ_0}{m}\frac{\partial F}{\partial\epsilon}\left(\frac{\partial}{\partial t} + \boldsymbol{V}.\boldsymbol{\nabla}_R\right)\tilde{\Phi} - \frac{qJ_0}{m\Omega}\boldsymbol{b}\times\boldsymbol{\nabla}_R\tilde{\Phi}.\boldsymbol{\nabla}_R F \\
&\quad + \frac{qJ_0}{m\Omega}\frac{\partial F}{\partial\epsilon}\boldsymbol{b}\times\boldsymbol{\nabla}_{\mathrm{R}}\tilde{\Phi}.\left[c_\| R\Omega'\frac{B_\phi}{B}\boldsymbol{\nabla}\psi\right] \qquad (7)
\end{aligned}
$$

We write the equilibrium distribution function as $F = A(\psi)e^{-m\epsilon/T(\psi)}$, where the velocity space variable corresponding to energy ϵ is given by:

$$
\epsilon = \frac{c_\|^2}{2} + \frac{c_\perp^2}{2} - \frac{V^2}{2} + \frac{q\phi_0(\psi)}{m}
$$

Now in our chosen ordering the centrifugal potential energy $V^2 \sim O(\varepsilon^2 v_t^2)$, and its radial derivative $m(V^2)' \sim O(\varepsilon T')$. Thus to leading order $\epsilon = c_\|^2/2 + c_\perp^2/2 + q\phi_0(\psi)/m$, and all flow terms can be dropped from the second term on the RHS, which is proportional to $\boldsymbol{\nabla}_R F$ and is the usual linear drive term in gyrokinetics. (This term corresponds to the second term on the RHS of equation (5)). The final term in equation (7) reduces to

$$
-\frac{mc_\| R\Omega'}{T}\frac{B_\phi}{B}F\frac{J_0}{B}\boldsymbol{b}\times\boldsymbol{\nabla}_{\mathrm{R}}\tilde{\Phi}.\boldsymbol{\nabla}\psi
$$

and corresponds to T_R in equation (4). We therefore find that applying these flow orderings to Artun and Tang's formulation of gyrokinetics with toroidal flows, confirms the validity of the simpler approach of section 1.1 that yielded equation (5).

1.3 Implementation of the Flow Shear Terms in GS2

Firstly we consider the time dependent eikonal arising from all terms in equation (2) where $\boldsymbol{U}.\boldsymbol{\nabla}$ acts on perturbation quantities A:

$$
\begin{aligned}
\boldsymbol{U}.\boldsymbol{\nabla}A &= R\Omega^x x\boldsymbol{e_\phi}.\boldsymbol{\nabla}\tilde{A} \qquad \text{where x is the radial variable in GS2} \\
&= in_0\tilde{A}R\Omega^x x\boldsymbol{e_\phi}.\boldsymbol{\nabla}\alpha \qquad \text{since } \tilde{A} = \tilde{A}(\theta)e^{in_0(\alpha+q\theta_0)}
\end{aligned}
$$

$$
\begin{aligned}
&= i n_0 \tilde{A} \Omega^x x \quad \text{since } \boldsymbol{e_\phi}.\boldsymbol{\nabla}\alpha = \frac{1}{R} \\
&= i \left(\frac{n_0 \rho^{\mathrm{ref}}}{a} \frac{d\rho_n}{d\psi_N} \right) \tilde{A}^{\mathrm{GS2}} \frac{d\psi_N}{d\rho_n} \Omega^x x \\
\Rightarrow \boldsymbol{U}.\boldsymbol{\nabla} A &= i k_y^{\mathrm{GS2}} \tilde{A}^{\mathrm{GS2}} \frac{d\psi_N}{d\rho_n} \Omega^x x \quad \text{using } k_y^{\mathrm{GS2}} = \frac{n_0 \rho^{\mathrm{ref}}}{a} \frac{d\rho_n}{d\psi_N}
\end{aligned} \tag{8}
$$

where we have obtained k_y^{GS2} from line 3 of page 5 in [1], $\psi_N = \psi/(B_0 a^2)$, a is the plasma half diameter in the equatorial midplane and B_0 is the reference magnetic field strength. To proceed in GS2 we require the definition of the GS2 radial coordinate x, which for general toroidal geometry is given in equation (6) of [2]:

$$
\begin{aligned}
x &= (\rho_n - \rho_{n0}) \frac{q_0}{\rho_{n0}} \frac{d\psi_N}{d\rho_n} \left(\frac{a}{\rho^{\mathrm{ref}}} \right) \\
\frac{dx}{d\rho_n} &= \frac{q_0}{\rho_{n0}} \frac{d\psi_N}{d\rho_n} \frac{a}{\rho^{\mathrm{ref}}}
\end{aligned} \tag{9}
$$

where ρ_n is the flux label, having the value ρ_{n0} at the centre of the box. The definition of x allows us to write:

$$
\begin{aligned}
\boldsymbol{U}.\boldsymbol{\nabla}\tilde{A} &= i k_y^{\mathrm{GS2}} \tilde{A}^{\mathrm{GS2}} \frac{d\psi_N}{d\rho_n} \frac{d\Omega}{d\rho_n} \frac{d\rho_n}{dx} x \\
&= i k_y^{\mathrm{GS2}} \tilde{A}^{\mathrm{GS2}} \left(\frac{\rho_{n0}}{q_0} \frac{d\Omega^{\mathrm{GS2}}}{d\rho_n} \right) \left(\frac{\rho^{\mathrm{ref}} v_t^{\mathrm{ref}}}{a^2} \right) x \quad \text{using (9) and } \Omega = \Omega^{\mathrm{GS2}} \frac{v_t^{\mathrm{ref}}}{a} \\
\Rightarrow \frac{\boldsymbol{U}.\boldsymbol{\nabla}\tilde{A}}{\left(\rho^{\mathrm{ref}} v_t^{\mathrm{ref}}/a^2\right)} &= i k_y^{\mathrm{GS2}} \tilde{A}^{\mathrm{GS2}} \left(\frac{\rho_{n0}}{q_0} \frac{d\Omega^{\mathrm{GS2}}}{d\rho_n} \right) x
\end{aligned} \tag{10}
$$

where the factor $\rho^{\mathrm{ref}} v_t^{\mathrm{ref}}/a^2$ can be cancelled from the gyrokinetic equation as it appears in all terms. GS2 uses coordinates x, y and θ in a stationary LAB frame, with x and y perpendicular to the magnetic field (x is a radial flux surface label and $y = \phi - q(x)(\theta - \theta_0)$ labels the field line), and θ is a poloidal angle that labels distance along the field direction. GS2 represents all perturbations in the following form:

$$
\tilde{A}(x, y, \theta, t) = \sum A_{k_x,k_y}(\theta, t) e^{ik_x x + ik_y y}
$$

Noting that:

$$
e^{-i\Gamma k_y t x} \frac{\partial}{\partial t} \sum A_{k_x,k_y}(\theta, t) e^{ik_x x + i\Gamma k_y t x + ik_y y} = \left(\frac{\partial}{\partial t} + i\Gamma k_y x \right) A \tag{11}
$$

allows the $\boldsymbol{U}.\boldsymbol{\nabla}$ terms to be absorbed directly into the eikonal, evolving the radial wavenumber of all perturbations in time according to $k_x(t) = k_x(0) + \Gamma k_y t$. Inside GS2 the evolving eikonal is computed using:

$$
k_x^{\mathrm{GS2}}(t) = k_x^{\mathrm{GS2}}(0) + \mathrm{GEXB}\; k_y^{\mathrm{GS2}} t^{\mathrm{GS2}}
$$

where t^{GS2} is GS2's internally normalised time (correponding to the timestep variable code_dt) which is ALWAYS given by

$$
t^{\mathrm{GS2}} = \frac{\sqrt{\frac{2T^{\mathrm{ref}}}{m^{\mathrm{ref}}}}\, t}{a}.
$$

The parameter GEXB is obtained from physical quantities using (10) to give:

$$
\mathrm{GEXB} = \frac{\rho}{q} \frac{d\Omega^{\mathrm{GS2}}}{d\rho_n} = \frac{\rho}{q} \frac{d\Omega}{d\rho_n} \frac{a m^{\mathrm{ref}}}{2T^{\mathrm{ref}}} \tag{12}
$$

where Ω is a real toroidal angular frequency in rads^{-1}, ρ_n is the GS2 normalised flux surface label (minor radius, square root toroidal flux, or poloidal flux), $L^{\mathrm{ref}} = a$ is the reference equilibrium length scale equal to the minor radius (or half midplane diameter), and v_t^{ref} is the reference thermal velocity. In the ballooning representation used for linear GS2 simulations, $k_x = k_{x0} + \hat{s} k_y (\theta - \theta_0)$ *(NB there is subtlety to be resolved here as $\hat{s}$ has a specific meaning that may differ from GS2's choice which may depend on the choice of GS2 radial variable)*, and the eikonal time dependence can be modelled with:

$$
\frac{d\theta_0}{dt^{\mathrm{GS2}}} = -\frac{\frac{dk_x^{\mathrm{GS2}}}{dt^{\mathrm{GS2}}}}{\hat{s} k_y^{\mathrm{GS2}}} = -\frac{\mathrm{GEXB}}{\hat{s}} \tag{13}
$$

$$\begin{aligned} \frac{d\theta_0}{dt} &= -\frac{\frac{d\Omega}{d\rho_n}}{\frac{dq}{d\rho_n}} \quad \text{from (12)} \\ \Rightarrow \frac{d\theta_0}{dt} &= -\frac{d\Omega}{dq} \end{aligned} \tag{14}$$

The flow shearing rate parameter γ_E (see Kinsey, Waltz and Candy, Phys Plasmas **12**, 062302, (2005)) is defined:

$$\gamma_E = \frac{r}{q}\frac{\partial(qv_E/r)}{\partial r} = \frac{\rho_{\phi n}}{q}\frac{\partial\Omega}{\partial\rho_{\phi n}} \quad \text{as } v_E = \frac{RB_p}{B}\Phi' \sim \frac{r}{q}\Omega(\psi) \ \text{ and where } r = \rho_{\phi n} \tag{15}$$

$$\Rightarrow \gamma_E^{\text{GS2}} = \left(\frac{\rho_{\phi n}}{\rho_n}\frac{d\rho_n}{d\rho_{\phi n}}\right)\frac{\rho_n}{q}\frac{d\Omega^{\text{GS2}}}{d\rho_n} = \left(\frac{\rho_{\phi n}}{\rho_n}\frac{d\rho_n}{d\rho_{\phi n}}\right)\text{GEXB} \tag{16}$$

Before considering the additional source term that arises with flow shear on the RHS of the gyrokinetic equation, it is helpful to understand how GS2 handles the usual linear drive terms L_D (eg that arising from the density gradient):

$$\begin{aligned} L_D &= -\frac{qJ_0}{m\Omega}\boldsymbol{b}\times\boldsymbol{\nabla}\tilde{\Phi}.\boldsymbol{\nabla}F \\ &= -in_0\frac{qJ_0}{m\Omega}\frac{\partial F}{\partial\psi}\tilde{\Phi}\boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\psi.\boldsymbol{b} \quad \text{since } \tilde{\phi} = \tilde{\phi}(\theta)e^{in_0\alpha} \\ &= -in_0J_0\frac{\partial F}{\partial\psi}\tilde{\Phi} \quad \text{since } \boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\psi.\boldsymbol{b} = B \\ &= -i\frac{n_0\rho^{\text{ref}}}{L^{\text{ref}}}\frac{T^{\text{ref}}}{q^{\text{ref}}}J_0\tilde{\Phi}^{\text{GS2}}\frac{\partial F}{\partial\psi} \quad \text{using } \tilde{\phi} = \frac{\rho^{\text{ref}}}{L^{\text{ref}}}\frac{T^{\text{ref}}\tilde{\phi}^{\text{GS2}}}{q^{\text{ref}}} \\ &= -i\frac{n_0\rho^{\text{ref}}}{L^{\text{ref}}}\frac{d\rho_n}{d\psi_n}\frac{T^{\text{ref}}}{q^{\text{ref}}B^{\text{ref}}L^{\text{ref}2}}J_0\tilde{\Phi}^{\text{GS2}}\frac{\partial F}{\partial\rho_n} \quad \text{using } \frac{dF}{d\psi} = \frac{1}{B^{\text{ref}}L^{\text{ref}2}}\frac{d\rho_n}{d\psi_n}\frac{dF}{d\rho_n} \\ &= -i\frac{n_0\rho^{\text{ref}}}{L^{\text{ref}}}\frac{d\rho_n}{d\psi_n}\frac{\rho^{\text{ref}}v_t^{\text{ref}}}{L^{\text{ref}2}}J_0\tilde{\Phi}^{\text{GS2}}\frac{\partial F}{\partial\rho_n} \quad \text{using } \frac{T^{\text{ref}}}{q^{\text{ref}}B^{\text{ref}}} = \rho^{\text{ref}}v_t^{\text{ref}} \\ &= i\left(\frac{n_0\rho^{\text{ref}}}{L^{\text{ref}}}\frac{d\rho_n}{d\psi_n}\right)J_0\tilde{\Phi}^{\text{GS2}}\left[-\frac{1}{F}\frac{\partial F}{\partial\rho_n}\right]\left(\frac{\rho^{\text{ref}}v_t^{\text{ref}}}{L^{\text{ref}2}}\right)F \\ \Rightarrow L_D &= ik_y^{\text{GS2}}J_0\tilde{\Phi}^{\text{GS2}}\left[-\frac{1}{F}\frac{\partial F}{\partial\rho_n}\right]\left(\frac{\rho^{\text{ref}}v_t^{\text{ref}}}{L^{\text{ref}2}}\right)F \quad \text{where } \ k_y^{\text{GS2}} = \frac{n_0\rho^{\text{ref}}}{L^{\text{ref}}}\frac{d\rho_n}{d\psi_n} \end{aligned} \tag{17}$$

The term in square brackets is the normalised source term that provides the linear drive for instability: e.g. a density gradient adds $(1/n)dn/d\rho_n$, and enters GS2 through the normalised input parameter fprim. We follow the same normalisation procedure for the additional source term T_R that arises with flow shear. Working from the final term in equation (7):

$$\begin{aligned} T_R &= \frac{qJ_0}{m\Omega}\frac{\partial F}{\partial\epsilon}\boldsymbol{b}\times\boldsymbol{\nabla}\tilde{\Phi}.\left[c_\parallel R\Omega'\frac{B_\phi}{B}\boldsymbol{\nabla}\psi\right] \\ &= in_0\frac{qJ_0}{m\Omega}\frac{\partial F}{\partial\epsilon}c_\parallel R\Omega'\frac{B_\phi}{B}\tilde{\Phi}\boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\psi.\boldsymbol{b} \quad \text{since } \tilde{\phi} = \tilde{\phi}(\theta)e^{in_0\alpha} \\ &= in_0J_0\frac{\partial F}{\partial\epsilon}c_\parallel R\Omega'\frac{B_\phi}{B}\tilde{\Phi} \quad \text{since } \boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\psi.\boldsymbol{b} = B \\ &= i\frac{n_0\rho^{\text{ref}}}{L^{\text{ref}}}\frac{T^{\text{ref}}}{q^{\text{ref}}}J_0\tilde{\Phi}^{\text{GS2}}\frac{\partial F}{\partial\epsilon}c_\parallel R\Omega'\frac{B_\phi}{B} \quad \text{using } \tilde{\phi} = \frac{\rho^{\text{ref}}}{L^{\text{ref}}}\frac{T^{\text{ref}}\tilde{\phi}^{\text{GS2}}}{q^{\text{ref}}} \\ &= i\frac{n_0\rho^{\text{ref}}}{L^{\text{ref}}}\frac{d\rho_n}{d\psi_n}\frac{T^{\text{ref}}}{q^{\text{ref}}B^{\text{ref}}L^{\text{ref}2}}J_0\tilde{\Phi}^{\text{GS2}}\frac{\partial F}{\partial\epsilon}c_\parallel R\frac{B_\phi}{B}\frac{d\Omega}{d\rho_n} \quad \text{using } \frac{d\Omega}{d\psi} = \frac{1}{B^{\text{ref}}L^{\text{ref}2}}\frac{d\rho_n}{d\psi_n}\frac{d\Omega}{d\rho_n} \\ &= i\frac{n_0\rho^{\text{ref}}}{L^{\text{ref}}}\frac{d\rho_n}{d\psi_n}\frac{\rho^{\text{ref}}v_t^{\text{ref}}}{L^{\text{ref}2}}J_0\tilde{\Phi}^{\text{GS2}}\frac{\partial F}{\partial\epsilon}c_\parallel R\frac{B_\phi}{B}\frac{d\Omega}{d\rho_n} \quad \text{using } \frac{T^{\text{ref}}}{q^{\text{ref}}B^{\text{ref}}} = \rho^{\text{ref}}v_t^{\text{ref}} \\ &= i\left(\frac{n_0\rho^{\text{ref}}}{L^{\text{ref}}}\frac{d\rho_n}{d\psi_n}\right)J_0\tilde{\Phi}^{\text{GS2}}\left[\frac{1}{F}\frac{\partial F}{\partial\epsilon}c_\parallel R\frac{B_\phi}{B}\frac{d\Omega}{d\rho_n}\right]\left(\frac{\rho^{\text{ref}}v_t^{\text{ref}}}{L^{\text{ref}2}}\right)F \\ \Rightarrow T_R &= ik_y^{\text{GS2}}J_0\tilde{\Phi}^{\text{GS2}}\left[\frac{-mc_\parallel RB_\phi}{TB}\frac{d\Omega}{d\rho_n}\right]\left(\frac{\rho^{\text{ref}}v_t^{\text{ref}}}{L^{\text{ref}2}}\right)F \end{aligned} \tag{18}$$

The additional source S that must be included in GS2 is the term in square brackets:

$$
\begin{aligned}
S &= -\frac{mc_\parallel RB_\phi}{TB}\frac{d\Omega}{d\rho_n} \\
&= -\frac{2c_\parallel RB_\phi}{v_{tj}^2 B}\frac{d\Omega}{d\rho_n} \quad \text{where } v_{tj} = \sqrt{\frac{2T_j T^{\mathrm{ref}}}{m_j m^{\mathrm{ref}}}} \\
&= -2\frac{c_\parallel}{v_{tj}}\frac{qRB_\phi}{\rho_{n0}B}\frac{1}{v_{tj}}\frac{\mathrm{GEXB}\sqrt{\frac{2T^{\mathrm{ref}}}{m^{\mathrm{ref}}}}L^{\mathrm{ref}}}{} \qquad \text{since } \frac{d\Omega}{d\rho_n} = \frac{q}{\rho_{n0}}\mathrm{GEXB}\frac{\sqrt{\frac{2T^{\mathrm{ref}}}{m^{\mathrm{ref}}}}}{L^{\mathrm{ref}}} \text{ from (12)} \\
\Rightarrow S &= -2\frac{c_\parallel}{v_{tj}}\frac{qR^{\mathrm{GS2}}B_\phi}{\rho_{n0}B}\sqrt{\frac{m_j}{T_j}}\mathrm{GEXB} \qquad (19)
\end{aligned}
$$

where T_j and m_j denote the temperature and mass of species j, normalised to the reference temperature and mass respectively.

1.4 Prescription for T_R used by Other Codes

My understanding is that other codes (GYRO and GKW) define their equilibrium distribution function as:

$$f_0^1 = n(x)\left(\frac{m}{2\pi T(x)}\right)^{1.5} e^{-m\left(\boldsymbol{v}-u_\parallel(x)\boldsymbol{b}\right)^2/2T(x)} \tag{20}$$

with the additional flux function equilibrium parameter $u_\parallel$ which contains mean flux parallel component of the equilibrium flow on the flux surface. This formulation of the equilibrium distribution function is of some concern for several reasons. Firstly it represents a sheared parallel flow in the equilibrium distribution function, which is not toroidal, and does not contribute to sheared perpendicular flow. Secondly the mean flow for this equilibrium is not divergence free. Thirdly, while modelling toroidal flow shear as a sheared parallel flow may be reasonable at large aspect ratio, at low aspect ratio the parallel direction is most different from the toroidal direction at the outboard midplane (ie in the bad curvature region). (Perhaps if the codes also include further terms for the equilibrium distribution function that include the sheared flow these concerns may be allayed.) The parameter $u_\parallel$ is set to zero, and $u_\parallel'$ is used to capture the contribution from the source term T_R. $u_\parallel'$ is chosen to cancel the poloidal component of equilibrium rotation arising from the equilibrium radial electric field and to give a resulting total flow in the toroidal direction. Including this parallel flow term can have a significant impact on the effectiveness of sheared flow stabilisation, as additional instability can be generated at large sheared parallel flows. It is therefore important that the term is implemented correctly. It is not entirely clear how this term has been implemented in other codes, but they can include many more terms from the equilibrium source than are in GS2 to include the neoclassical corrections to the leading order distribution function. The GS2 source on the right hand side includes only the set of linear terms in the perturbations acting on the leading order equilibrium Maxwellian distribution, while GKW can include all of the drifts acting on the Maxwellian. It is possible that what is appropriate for f_0^1 in one code may not be appropriate for another.

1.4.1 The Parallel Flow

A sketch to motivate the need for a parallel flow if the flow arising strictly from V_{E0} has been fully implemented is given below.

$$
\begin{aligned}
\boldsymbol{V_E} &= \frac{B\times\boldsymbol{\nabla}\Phi}{B^2} \text{ where } \boldsymbol{B} = \boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\phi + I\boldsymbol{\nabla}\phi \\
\Rightarrow \boldsymbol{V_E}.\boldsymbol{\nabla}\theta &= \frac{I\Phi'}{B^2}\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}\psi.\boldsymbol{\nabla}\theta \\
\Rightarrow \boldsymbol{V_E}.\boldsymbol{\nabla}\phi &= \Phi'\frac{(\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\phi)\times\boldsymbol{\nabla}\psi}{B^2}.\boldsymbol{\nabla}\phi = \Phi'\frac{|\boldsymbol{\nabla}\psi|^2|\boldsymbol{\nabla}\phi|^2}{B^2}. = \Phi'\frac{B_p^2}{B^2}.
\end{aligned}
$$

A parallel flow arises to cancel the poloidal component of $\boldsymbol{V_E}$:

$$\boldsymbol{V}_\parallel = \frac{U_\parallel^s\boldsymbol{B}}{B}$$

$$
\begin{aligned}
\Rightarrow \boldsymbol{V}_{\|}.\boldsymbol{\nabla}\theta &= \frac{U_{\|}^{s}}{B}\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\theta \\
\Rightarrow \text{ setting } U_{\|} &= \frac{I\Phi'}{B} \text{ cancels the theta component of the net flow} \\
\Rightarrow \boldsymbol{V}_{\|}.\boldsymbol{\nabla}\phi &= \frac{U_{\|}I}{R^2 B}. = \frac{B_\phi^2 \Phi'}{B^2}
\end{aligned}
$$

The net flow now lies in the toroidal direction:

$$
\left(\boldsymbol{V_E} + \boldsymbol{V}_{\|}\right).\boldsymbol{\nabla}\phi = \Phi'
$$

and the mean flow on a flux surface corresponds to rigid toroidal rotation with frequency $\Omega = \Phi'$, which is manifestly divergence free. It is to be noted that the magnitude of $U_{\|} = \frac{I\Phi'}{B}$ is not an exact flux function.

(Diamagnetic flows are species dependent fluid flows that arise from circulating orbits, and are given by:

$$
\begin{aligned}
\boldsymbol{V_D^s} &= \frac{\boldsymbol{B} \times \boldsymbol{\nabla} p_s}{n_s q_s B^2} \\
\Rightarrow \boldsymbol{V_D^s}.\boldsymbol{\nabla}\theta &= \frac{I p_s'}{n_s q_s B^2}\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi.\boldsymbol{\nabla}\theta \\
\Rightarrow \boldsymbol{V_D^s}.\boldsymbol{\nabla}\phi &= \frac{p_s'}{n_s q_s}\frac{B_p^2}{B^2}.
\end{aligned}
$$

These flows generate current but no mass flow.)

2 GS2 (November 2007)

This outlines CMR's understanding of how GS2 implements equilibrium flow shear. The gyrokinetic equation for the time evolution of the nonadiabatic perturbed distribution function g is given by:

$$
\frac{\partial g}{\partial t} = \hat{G}g + \boldsymbol{v}.\boldsymbol{\nabla} g
$$

where $\hat{G}$ denotes the usual gyrokinetic operator for the advance of g without equilibrium flow. With perpendicular flow shear only, $\boldsymbol{v} = v' x \boldsymbol{e_y}$. Numerically GS2 proceeds to solve for g by splitting the time advance into steps. Here we need only consider the time advance step due to the inclusion of flow shear, which looks like:

$$
\frac{\Delta g}{\Delta t} = v' x \frac{\partial g}{\partial y}
$$

GS2 uses coordinates x, y and θ in a stationary LAB frame, with x and y perpendicular to the magnetic field, and θ along the field direction. x is radial and y lies in the magnetic surface. GS2 represents all perturbations in the following form:

$$
A(x, y, \theta, t) = \sum A_{k_x,k_y}(\theta, t) e^{ik_x x + ik_y y}
$$

The flow shear advance step for g:

$$
\Delta g = \sum_{k_x,k_y} ik_y v' x \Delta t g_{k_x,k_y}(\theta, t) e^{ik_x x + ik_y y}
$$

and for infinitesimal Δt this can be written as:

$$
\Delta g = \sum_{k_x,k_y} \left(e^{ik_y v' x \Delta t} - 1\right) g_{k_x,k_y}(\theta, t) e^{ik_x x + ik_y y}
$$

Therefore the flow shear time advance step should yield:

$$
g^{t+\Delta t} = \sum_{k_x,k_y} g_{k_x,k_y}(\theta, t) e^{i\left(k_x + k_y v' \Delta t\right)x + ik_y y}
$$

which yields in terms of the Fourier components:

$$g_{k_x,k_y}(\theta, t + \Delta t) = g_{k_x - k_{x\ \mathrm{shift}},k_y}(\theta, t) \tag{21}$$

where $k_{x\ \mathrm{shift}} = k_y v' \Delta t, k_y$. With a continuous grid in k_x and infinitesimal Δt this would describe exactly equilibrium flow shear acting alone.

GS2 has a discrete grid in k_x with grid spacing Δk_x. The evolution of $k_{x\ \mathrm{shift}}$ is monitored in time, and when $k_{x\ \mathrm{shift}}$ exceeds Δk_x (or is it $\Delta k_x/2$) GS2 advects all perturbations in k_x using equation 21 to improve the GS2 approximation of the perturbations in the LAB frame. The perturbation can be computed at any point in time in the fixed GS2 LAB frame using:

$$A(x, y, \theta, t) = \sum A_{k_x,k_y}(\theta, t) e^{i(k_x + k_{x\ \mathrm{shift}})x + ik_y y}$$

but $k_{x\ \mathrm{shift}}$ is not included in the $\hat{G}$ time advance step for the reason that including it in this step would be prohibitively expensive computationally.

References

[1] W. Dorland and M. Kotschenreuther, Microinstabilities in Axisymmetric Configurations.

[2] D. J. Applegate, GS2 Coordinate Transformation for Visualisation, (2008).